

The Chermak-Delgado lattice of ZM-groups

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Abstract

In this note we prove that the Chermak-Delgado lattice of a ZM-group is a chain of length 0. A similar conclusion is obtained for all dihedral groups D_{2m} with $m \neq 4$.

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1 Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G . The *Chermak-Delgado measure* of a subgroup H of G is defined by

$$m_G(H) = |H||C_G(H)|.$$

Let

$$m(G) = \max\{m_G(H) \mid H \leq G\} \text{ and } \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a modular self-dual sublattice of $L(G)$, which is called the *Chermak-Delgado lattice* of G . It was first introduced by Chermak and Delgado [6], and revisited by Isaacs [8]. In the last years there has been a growing interest in understanding this lattice, especially for p -groups (see e.g. [1, 3, 4, 10]). The study can be naturally extended to nilpotent groups, since by [3] the Chermak-Delgado lattice of a direct product of finite groups decomposes as the direct product of the Chermak-Delgado lattices of the factors. Recall also two other important properties of the Chermak-Delgado lattice that will be used in our paper:

- if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the minimum subgroup $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak-Delgado subgroup* of G) is characteristic, abelian, and contains $Z(G)$.

In what follows we will focus on describing the Chermak-Delgado lattice of a ZM-group, that is a finite group with all Sylow subgroups cyclic. By [7] such a group is of type

$$\text{ZM}(m, n, r) = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle,$$

where the triple (m, n, r) satisfies the conditions

$$\gcd(m, n) = \gcd(m, r - 1) = 1 \text{ and } r^n \equiv 1 \pmod{m}.$$

It is clear that $|\text{ZM}(m, n, r)| = mn$ and $Z(\text{ZM}(m, n, r)) = \langle b^d \rangle$, where d is the multiplicative order of r modulo m , i.e.

$$d = \min\{k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod{m}\}.$$

The subgroups of $\text{ZM}(m, n, r)$ have been completely described in [5]. Set

$$L = \left\{ (m_1, n_1, s) \in \mathbb{N}^3 \mid m_1 \mid m, n_1 \mid n, s < m_1, m_1 \mid s \frac{r^{n_1} - 1}{r^{n_1} - 1} \right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\text{ZM}(m, n, r))$ of $\text{ZM}(m, n, r)$, namely the function that maps a triple $(m_1, n_1, s) \in L$ into the subgroup $H_{(m_1, n_1, s)}$ defined by

$$H_{(m_1, n_1, s)} = \bigcup_{k=1}^{\frac{n}{n_1}} \alpha(n_1, s)^k \langle a^{m_1} \rangle = \langle a^{m_1}, \alpha(n_1, s) \rangle,$$

where $\alpha(x, y) = b^x a^y$, for all $0 \leq x < n$ and $0 \leq y < m$. Note that:

- $|H_{(m_1, n_1, s)}| = \frac{mn}{m_1 n_1}$, for any s satisfying $(m_1, n_1, s) \in L$;
- $H_{(m_1, n_1, s)}$ is normal in $\text{ZM}(m, n, r)$ if and only if $m_1 \mid r^{n_1} - 1$ and $s = 0$ (see [9]);
- two subgroups of $\text{ZM}(m, n, r)$ are conjugate if and only if they have the same order (see [2]).

We are now able to give our main result.

Theorem 1. *Under the above notation, we have*

$$m(\text{ZM}(m, n, r)) = \frac{m^2 n^2}{d^2}.$$

Moreover, $\mathcal{CD}(\text{ZM}(m, n, r))$ is a chain of length 0, namely

$$\mathcal{CD}(\text{ZM}(m, n, r)) = \{H_{(1,d,0)}\}.$$

By taking an odd integer $m \geq 3$, $n = 2$ and $r = m - 1$ one obtains $d = 2$ and $\text{ZM}(m, n, r) \cong D_{2m}$. Thus, Theorem 1 leads to the following corollary.

Corollary 2. *If $m \geq 3$ is odd, then*

$$m(D_{2m}) = m^2.$$

Moreover, $\mathcal{CD}(D_{2m})$ is a chain of length 0, namely

$$\mathcal{CD}(D_{2m}) = \{\langle a \rangle\}.$$

Now, let us assume that $m = 2^k m'$ with $k \geq 1$ and m' odd. For $m' \geq 3$ the conclusion of Corollary 2 remains valid because $D_{2m} \cong \mathbb{Z}_2^k \times D_{2m'}$, and therefore

$$\mathcal{CD}(D_{2m}) \cong \mathcal{CD}(\mathbb{Z}_2^k) \times \mathcal{CD}(D_{2m'}) = \{\mathbb{Z}_2^k\} \times \mathcal{CD}(D_{2m'})$$

is a chain of length 0. We infer that the computation of $\mathcal{CD}(D_{2m})$ is completed by studying the case $m' = 1$, i.e. for

$$D_{2^{k+1}} = \langle a, b \mid a^{2^k} = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

We easily obtain

$$m(D_{2^{k+1}}) = 2^{2k}, \forall k \geq 2$$

and

$$\mathcal{CD}(D_{2^{k+1}}) = \begin{cases} \{D_8, \langle a \rangle, \langle a^2, b \rangle, \langle a^2, ab \rangle, \langle a^2 \rangle\}, & k = 2 \\ \{\langle a \rangle\}, & k \geq 3. \end{cases}$$

Finally, we remark that $\mathcal{CD}(G)$ is a chain of length 0 for large classes of groups G , such as abelian groups, ZM-groups or dihedral groups D_{2m} with $m \neq 4$. This leads to the following natural question.

Open problem. Which are the finite groups G such that $\mathcal{CD}(G) = \{M(G)\}$?

2 Proof of the main result

We start by proving two auxiliary results.

Lemma 3. *For every $(m_1, n_1, s) \in L$, we have*

$$m_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)}) = \frac{mn^2 \gcd(m, r^{n_1} - 1)}{m_1 n_1 \text{ord}_{\frac{m}{m_1}}(r)},$$

where $\text{ord}_{\frac{m}{m_1}}(r)$ denotes the multiplicative order of r modulo $\frac{m}{m_1}$.

Proof. First of all, we observe that under the notation in Section 1 we have

$$\alpha(x_1, y_1)\alpha(x_2, y_2) = \alpha(x_1 + x_2, r^{x_2}y_1 + y_2).$$

This implies that

$$\alpha(x, y)^k = b^{kx} a^{y^{\frac{r^{kx}-1}{r^x-1}}}, \text{ for all } k \in \mathbb{Z},$$

and so

$$\alpha(x, y)^{-1} = \alpha(-x, -r^{-x}y).$$

Then

$$\alpha(x, y)^{-1}\alpha(u, v)\alpha(x, y) = \alpha(u, -r^u y + r^x v + y).$$

Now, let $(m_1, n_1, s) \in L$. In order to compute $|C_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)})|$ we can assume $s = 0$, because $H_{(m_1,n_1,s)}$ and $H_{(m_1,n_1,0)}$ are conjugate. We obtain $\alpha(x, y) \in C_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,0)})$ if and only if

$$\alpha(x, y)^{-1}\alpha(0, m_1)\alpha(x, y) = \alpha(0, m_1) \text{ and } \alpha(x, y)^{-1}\alpha(n_1, 0)\alpha(x, y) = \alpha(n_1, 0),$$

which means

$$\alpha(0, r^x m_1) = \alpha(0, m_1) \text{ and } \alpha(n_1, -r^{n_1}y + y) = \alpha(n_1, 0),$$

i.e.

$$\frac{m}{m_1} \mid r^x - 1 \text{ and } m \mid y(r^{n_1} - 1).$$

Clearly, these relations are equivalent respectively with

$$\text{ord}_{\frac{m}{m_1}}(r) \mid x \text{ and } \frac{m}{\gcd(m, r^{n_1} - 1)} \mid y,$$

that is $C_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,0)}) = H_{(m'_1,n'_1,0)}$, where

$$m'_1 = \frac{m}{\gcd(m, r^{n_1} - 1)} \text{ and } n'_1 = \text{ord}_{\frac{m}{m_1}}(r).$$

In particular, we have

$$|C_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)})| = |H_{(m'_1,n'_1,0)}| = \frac{mn}{m'_1 n'_1} = \frac{n \gcd(m, r^{n_1} - 1)}{\text{ord}_{\frac{m}{m_1}}(r)},$$

and consequently

$$m_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)}) = m_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,0)}) = \frac{mn^2 \gcd(m, r^{n_1} - 1)}{m_1 n_1 \text{ord}_{\frac{m}{m_1}}(r)},$$

as desired. \square

The computation of the maximum value of $m_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)})$ when $m_1 \mid m$ and $n_1 \mid n$ is difficult by using only Lemma 3. In order to do this the following step is crucial.

Lemma 4. *The Chermak-Delgado subgroup of $\text{ZM}(m, n, r)$ is $H_{(1,d,0)}$.*

Proof. Let $H_{(m_0,n_0,s)} = M(\text{ZM}(m, n, r))$. Then the triple (m_0, n_0, s) satisfies the following conditions:

- a) $m_0 \mid r^{n_0} - 1$ and $s = 0$, because $H_{(m_0,n_0,s)}$ is normal in $\text{ZM}(m, n, r)$;
- b) $\frac{m}{m_0} \mid r^{n_0} - 1$, because $H_{(m_0,n_0,s)}$ is abelian;
- c) $n_0 \mid d$, because $Z(\text{ZM}(m, n, r)) \subseteq H_{(m_0,n_0,s)}$.

Also, if $H_{(m'_0,n'_0,0)} = C_{\text{ZM}(m,n,r)}(H_{(m_0,n_0,s)})$, then we have

$$m'_0 = \frac{m}{\gcd(m, r^{n_0} - 1)} \text{ and } n'_0 = \text{ord}_{\frac{m}{m_0}}(r)$$

by Lemma 3. Since $C_{\text{ZM}(m,n,r)}(H_{(m'_0,n'_0,0)}) = H_{(m_0,n_0,s)}$ we infer that

$$\frac{m}{m_0} = \gcd(m, r^{n'_0} - 1) \text{ and } n_0 = \text{ord}_{\gcd(m, r^{n'_0} - 1)}(r).$$

Let $d = n'_0 \alpha$, where $\alpha \in \mathbb{N}^*$. Then the condition

$$r^d - 1 = (r^{n'_0} - 1) \sum_{i=0}^{\alpha-1} r^{n'_0 i} \equiv 0 \pmod{m},$$

implies

$$\sum_{i=0}^{\alpha-1} r^{n'_0 i} \equiv 0 \pmod{m_0}.$$

Assume that $\gcd(m_0, \frac{m}{m_0}) \neq 1$ and take a prime p dividing $\gcd(m_0, \frac{m}{m_0})$. Since $p \mid \frac{m}{m_0}$, it follows that

$$r^{n'_0} \equiv 1 \pmod{p},$$

and therefore

$$\sum_{i=0}^{\alpha-1} r^{n'_0 i} \equiv \alpha \pmod{p}.$$

On the other hand, by $p \mid m_0$ we have

$$\sum_{i=0}^{\alpha-1} r^{n'_0 i} \equiv 0 \pmod{p}.$$

Then $p \mid \alpha$, implying that $p \mid n$. One obtains $p \mid \gcd(m, n)$, a contradiction. Consequently,

$$\gcd(m_0, \frac{m}{m_0}) = 1.$$

It is now clear that the conditions a) and b) imply $m \mid r^{n_0} - 1$, i.e. $d \mid n_0$, which together with the condition c) leads to

$$n_0 = d.$$

Next, we will show that $m_0 = 1$. Assume $m_0 \neq 1$ and denote $\beta = \text{ord}_{m_0}(r)$. Since both m_0 and $\frac{m}{m_0}$ divide $r^{\beta n'_0} - 1$, we infer that $m \mid r^{\beta n'_0} - 1$, and so $d \mid \beta n'_0$. Then

$$d \leq \beta n'_0 \leq \varphi(m_0) n'_0 < m_0 n'_0,$$

which implies

$$m_{\text{ZM}(m,n,r)}(H_{(m_0,n_0,s)}) = \frac{m^2 n^2}{d m_0 n'_0} < \frac{m^2 n^2}{d^2} = m_{\text{ZM}(m,n,r)}(H_{(1,d,0)}),$$

contradicting the maximality of $m_{\text{ZM}(m,n,r)}(H_{(m_0,n_0,s)})$. Hence $m_0 = 1$, as desired. \square

We have now all ingredients to prove our main theorem.

Proof of Theorem 1. The equality

$$m(\text{ZM}(m, n, r)) = \frac{m^2 n^2}{d^2}$$

follows by Lemma 4. Let $H_{(m_1,n_1,s)} \in \mathcal{CD}(\text{ZM}(m, n, r))$. Then

$$m_{\text{ZM}(m,n,r)}(H_{(m_1,n_1,s)}) = \frac{mn^2 \gcd(m, r^{n_1} - 1)}{m_1 n_1 \text{ord}_{\frac{m}{m_1}}(r)} = \frac{m^2 n^2}{d^2},$$

or equivalently

$$d^2 \gcd(m, r^{n_1} - 1) = mm_1 n_1 \text{ord}_{\frac{m}{m_1}}(r).$$

Since $\gcd(m, r^{n_1} - 1)$, m and m_1 are divisors of m , while d , n_1 and $\text{ord}_{\frac{m}{m_1}}(r)$ are divisors of n , and $\gcd(m, n) = 1$, we infer that

$$\gcd(m, r^{n_1} - 1) = mm_1 \text{ and } d^2 = n_1 \text{ord}_{\frac{m}{m_1}}(r).$$

Clearly, these equalities imply

$$m_1 = 1 \text{ and } n_1 = d,$$

and by $s < m_0$ we obtain

$$s = 0.$$

Hence $H_{(m_1,n_1,s)} = H_{(1,d,0)}$, completing the proof. \square

References

- [1] L. An, J.P. Brennan, H. Qu and E. Wilcox, *Chermak-Delgado lattice extension theorems*, Comm. Algebra **43** (2015), 2201-2213.
- [2] R. Brandl, G. Cutolo and S. Rinauro, *Posets of subgroups of groups and distributivity*, Boll. U.M.I. **9-A** (1995), 217-223.
- [3] B. Brewster and E. Wilcox, *Some groups with computable Chermak-Delgado lattices*, Bull. Aus. Math. Soc. **86** (2012), 29-40.

- [4] B. Brewster, P. Hauck and E. Wilcox, *Groups whose Chermak-Delgado lattice is a chain*, J. Group Theory **17** (2014), 253-279.
- [5] W.C. Calhoun, *Counting subgroups of some finite groups*, Amer. Math. Monthly **94** (1987), 54-59.
- [6] A. Chermak and A. Delgado, *A measuring argument for finite groups*, Proc. AMS **107** (1989), 907-914.
- [7] B. Huppert, *Endliche Gruppen*, I, Springer Verlag, Berlin, 1967.
- [8] I.M. Isaacs, *Finite group theory*, Amer. Math. Soc., Providence, R.I., 2008.
- [9] M. Tărnăuceanu, *The normal subgroup structure of ZM-groups*, Ann. Mat. Pura Appl. **193** (2014), 1085-1088.
- [10] E. Wilcox, *Exploring the Chermak-Delgado lattice*, Math. Magazine **89** (2016), 38-44.

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